CLNS 97/1476

Quantum-type Coherence as a Combination of Symmetry and Semantics.

Yuri F. Orlov

February 1, 2008

Floyd R. Newman Laboratory of Nuclear Studies
Cornell University, Ithaca, New York 14853 USA

Abstract

It is shown that quantum-type coherence, leading to indeterminism and interference of probabilities, may in principle exist in the absence of the Planck constant and a Hamiltonian. Such coherence is a combined effect of a symmetry (not necessary physical) and semantics. The crucial condition is that symmetries should apply to logical statements about observables. A theoretical example of a non-quantum system with quantum-type properties is analysed.

Coherence, a cornerstone of quantum mechanics, is considered to be a result of the quantization of action. However, we will show that "quantum-type coherence," as we will call it, does not depend on the existence of the Planck constant (although its concrete manifestations do). In our two examples, E_1 and E_2, analysed below, such coherence appears in the presence of:(I) An ordered set of mutually exclusive objects, with numerical values $\xi, \xi \in$A, where A is an affine space (a space with no fixed origin); ξ is a particle coordinate in E_1, and an interpretation of a given situation in E_2. (II) Another ordered set of mutually exclusive objects, with numerical values χ defined on set (I) *as a whole*; χ is a value of the particle momentum in E_1, and in E_2 the ordinal number of a logical statement in a set of mutually exclusive statements describing the situation. Conditions(I) and (II) imply the existence of a symmetry. It is shown that when symmetries apply to logical statements about objects instead of objects "per se," the following semantic problems arise: in E_1, the problem of expressing the truth values of logical statements about objects in the second set, in terms of the truth values of logical statements about objects in the first set; and in E_2, the problem of expressing

the truth values of statements in one interpretation when the truth values of the same statements in another interpretation are given. Inexpressibility is therefore a combined effect of symmetry and semantics—both irrelevant to $\hbar$. This effect, fundamental for quantum-type coherence, leads to indeterminism and interference of probabilities.

We will first delineate the border between the symmetry-semantic part (without $\hbar$) and the quantum part (with $\hbar$) of the quantum mechanical formalism, using an example of a single, zero-spin particle. Then we will analyse a non-quantum system having all typical features of quantum-type coherence. Its analysis provides a theoretical basis for searching for systems, neither classical nor quantum, in which quantum-type interference can be observed. Observation of such systems, interesting in itself, may indirectly clarify our understanding of the structure of quantum mechanics and the origins of quantization.

E_1.*A quantum example.* We will construct the quantum mechanics formalism of a single, zero-spin particle, introducing assumptions step-by-step so as to make clear exactly at which point in our construction $\hbar$ is needed. We will tag our assumptions with Greek letters.

Our fundamental assumption about what makes mechanics "quantum" is that (α) the statement Λ_{p_0} : "$p = p_0$" about a particle momentum p (a translational invariant in the coordinate q-space) should itself be an invariant of the translational symmetry in the same space. Such a requirement makes sense only if Λ_{p_0}, a logical statement, is simultaneously a function of coordinates, such that it does not depend on transformations $q \to q + \delta q$. This is possible only if Λ_{p_0}, as a function of coordinates, either does not depend on q at all, or depends only on differences between coordinates. We suppose the latter, $(\beta)\Lambda_{p_0}(q, q') = \Lambda_{p_0}(q' - q)$. In our next step, $\Lambda_{p_0}(q, q')$ is assumed to be a matrix in q-space; and since Λ_{p_0} is a logical statement, we can use logic to calculate it. Consider the logical equivalence, $\Lambda_{p_0} \wedge \Lambda_{p_0} \sim \Lambda_{p_0}$. The natural assumption is that (γ): the logical conjunction in this equivalence should be represented by the matrix product, and the logical equivalence by the equation:

$$\int \Lambda_{p_0}(q' - s)\Lambda_{p_0}(s - q)ds = \Lambda_{p_0}(q' - q). \tag{1}$$

The solution of this equation is $\Lambda_{p_0} = L^{-1} exp 2\pi i(q' - q)\lambda^{-1}(p_0)$, where the wave length $\lambda(p_0)$ is an unknown function of p_0, and $L \equiv \int dq$. The eigenvectors of this matrix (up to normalization constants) are $\psi_p(q) = exp 2\pi i q/\lambda(p), \lambda(p_1) \neq \lambda(p_2)$ for $p_1 \neq p_2$. Thus, we already have—without $\hbar$—the correct wave functions and density matrices, though the dependence of λ on p remains unknown. To be consistent, we now assume that a statement "$q = q_0$" about particle coordinate q should also be represented in q-space by a matrix. Obviously this matrix must be $(\delta) : \Lambda_{q_0}(q, q') = K\delta(q' - q_0)\delta(q - q_0)$, with K constant. The eigenvectors of Λ_{q_0} are $\psi_{q_i}(q) = \delta(q - q_i)$.

Matrices $\Lambda_{q_0}(q,q')$ and $\Lambda_{p_0}(q,q')$ do not commute; their commutator is not proportional to $\hbar$. While Λ_{q_0} is a statement about the exact location of the particle, statement Λ_{p_0} expresses *by its own symmetric structure* the uncertainty of that location, and is defined by the momentum value, p_0, which, according to the meaning of translational invariance, relates to q-space as a whole. If, now, Λ_{p_0} is true, i.e., is the correct description of the state of the particle, and the question is whether Λ_{q_0} is true, the answer can be at best probabilistic, since the truth of Λ_{p_0} is inexpressible in terms of the truth of any Λ_{q_i}. Here we have the probability that the conjunction $\Lambda_{p_0} \wedge \Lambda_{q_0}$ is true. Since, according to (γ), conjunctions are represented by matrix products, and since the probability should be a translational invariant and be independent of the order of matrices, the only correct formula is:

$$w(p_0|q_0) = \int \Lambda_{p_0}(q,q')\Lambda_{q_0}(q',q)dqdq' = K/L. \tag{2}$$

Only after this step in constructing the quantum formalism need we introduce $\hbar$. The physical part of the quantum formalism is then defined by the introduction of Hamiltonian, canonical transformations, etc.

E_2. *A non-quantum example.* Consider a situation open to interpretation and describable by different logical connections among n independent logical statements, $\lambda_i, i = 1,2,...,n$. According to the classical logic of propositions, every such description can be represented by a disjunction of mutually exclusive conjunctions. There are $N = 2^n$ such conjunctions, which can be enumerated as $\Lambda_k, k = 1,2...,N; N \geq 2$. One, and only one of them, can be true. By definition, a *certain* (i.e., not uncertain) interpretation $I^{(s)}, I^{(s)} = I^{(s)}(k)$, is the following function of the integer $k, k = 1,2,...N$: if some Λ_l is defined as a true statement-conjunction, then $I^{(s)}(k) = \delta_{kl}$. Inversely, if $I^{(s)}(k) = \delta_{kl}$, then in this interpretation Λ_l is true. Two interpretations are identical if the corresponding functions are identical. There are only N non-identical certain interpretations, each defining which *one* of the N conjunctions is true. N non-identical certain interpretations can be transformed into each other by permutations, described by the finite table. That table can be considered as an algorithm defining the truth values of the conjunctions in all certain interpretations, when the truth values of the conjunctions in one of them are defined.

Now we will extend our concept of interpretation beyond classical logic by introducing the following two conditions: (a)(symmetry) There is no correct interpretation a priori: any statement-conjunction may be considered true. Such a choice defines a *correct* interpretation. Once an interpretation is considered correct, N different certain interpretations can be generated by permutations of truth values. By definition, these certain interpretations define the meaning of the truth values of N conjunctions. (b) If there are two interpretations, $I^{(s)}(k)$ and $I^{(s')}(k)$, then the difference between them

is measured by a real number, θ; all values of θ inside an interval $\theta_{min} \leq \theta \leq \theta_{max}$ are permitted; and if $I^{(s)}(k)$ is considered correct, then $\theta = s' - s$.

More formally, all interpretations are now points in an affine space $A^1, s \in A^1$, and $\vec{\theta}$ is a vector in R^1 vector space of real numbers, $\theta \in R^1$. The group R^1 acts on A^1 as the continuous group of parallel displacements. Points of our affine space are *functions* defined in their own discrete spaces, each containing N points, $N \geq 2$. We will show that such a system possesses the properties of quantum-type coherence: indeterminism, interference of probabilities, and the possibility of introducing wave functions, though none of our assumptions depends on $\hbar$. The reason, qualitatively, is that now we have a continuum of interpretations; but when we define a meaning of truth values of N conjunctions that is equal for all interpretations, only N interpretations (which, according to (a), can be chosen arbitrarily) can be certain. In all other interpretations, truth values of conjunctions are not certain.

Theorem 1. On the existence of inexpressibility.

If conditions (a) and (b) are met, then either all interpretations are identical or there does not exist any algorithm, defined for θ in the interval $[\theta_{min}, \theta_{max}]$, to calculate the truth values of statements in an interpretation $I^{(s')}(k)$ when some other interpretation, $I^{(s)}(k)$, is considered correct

Proof. Let statements $\Lambda_l, l = 1, 2, ..., N$, in some interpretation,$I^{(0)}(l)$, that is considered correct, possess given truth values, and let not all interpretations be identical. Then there exists some θ that defines an interpretation, $I^{(\theta)}(k)$, different from $I^{(0)}(k)$. This means that at least one of the statements in $I^{(\theta)}$, let it be Λ_i, does not possess the same truth value as Λ_i in $I^{(0)}$. Assume that there exists an algorithm mapping the truth values of statements in $I^{(0)}$ onto the truth values of statements in $I^{(\theta)}$. Since two different distributions of truth values among N statements are permutations of each other, any assumed algorithm should define the operation of a permutation, which should depend on θ. Consider the parameter $\delta\theta = \theta / N!$ connecting any two interpretations with $s' - s = \delta\theta$. According to our assumption, this parameter defines some permutation of truth values. Consider *N!* such consecutive permutations, beginning from the given distribution of truth values in the initial interpretation, $I^{(0)}$. On one hand, *N!* identical permutations give us the same final distribution of truth values as the initial one. But, on the other hand, since $\delta\theta \cdot N! = \theta$, we will arrive at interpretations $I^{(\theta)}$, which is not identical to the initial interpretation. The contradiction means that our assumption about the existence of a mapping algorithm dependent on θ was wrong. It also means that not all interpretations can be certain. The truth values of N conjunctions in uncertain interpretations cannot be expressed in terms of the truth values of these conjunctions in certain interpretations. □

Thus, under the conditions satisfying Theorem 1, the problem of expressing the

truth values of statements in an arbitrary interpretation, when the truth values of statements in another, certain interpretation are given, is insoluble. Therefore, in the general case, if a question arises whether a particular statement in an interpretation is true when the truth values of statements in another interpretation are given, the answer is unpredictable. If the concept of probability applies to such a system, then the probability of a certain answer can depend only on the parameter defining the difference between interpretations, θ. This leads to

Theorem 2. Under the conditions of Theorem 1, the probabilities $p(\theta)$ of complex events do not obey classical rules, and have interference terms.

*Proof.*It is sufficient to prove the theorem for the simplest case N=2, in which there are only two mutually exclusive statements in every interpretation, Λ and $\overline{\Lambda}$; the latter is the negation of the first. Consider three pairs of interpretations, $\left(I^{(0)}, I^{(\theta)}\right)\left(, I^{(\theta)}, I^{(\theta+\vartheta)}\right)$, and $\left(I^{(0)}, I^{(\theta+\vartheta)}\right)$. We will label s the statement, either Λ or $\bar{\Lambda}$, whose truth values are interpreted in the (maybe uncertain) interpretation $I^{(s)}(k), k=1,2;\ \Lambda_1 \equiv \Lambda, \Lambda_2 \equiv \overline{\Lambda}$. Probabilities of the answer "yes" to the questions "Is $\Lambda^{(r)}$ (or $\bar{\Lambda}^{(r)}$) true, if $\Lambda^{(s)}$ is true?" will be denoted as $p(s,r) \equiv p(r-s)$, and $p(s,\bar{r}) = 1 - p(s,r)$; and the probabilities of the answer "yes" to the same questions but with $\bar{\Lambda}^{(s)}$ instead of $\Lambda^{(s)}$ will be denoted as $p(\bar{s},r) = 1 - p(s,r)$, and $p(\bar{s},\bar{r}) = p(s,r)$. The equalities follow from the fact that in any interpretation, certain or not, one of the statements is true and the others are false, because the disjunction

$$\Lambda_1 \vee \Lambda_2 \vee ... \vee \Lambda_N \equiv T \tag{3}$$

is an invariant (a tautology) independent of the choice of interpretation.

Given questions corresponding to the aforementioned three pairs of interpretations, such that either $\Lambda^{(s)}$ or $\bar{\Lambda}^{(s)}$ is true in interpretation $I^{(s)}$ of every pair, the classical probability formula for the answers should be: $p(0,\theta+\vartheta) = p(0,\theta)p(\theta,\theta+\vartheta) + p(0,\bar{\theta})p(\bar{\theta},\theta+\vartheta)$, which can be rewritten as

$$p(\theta+\vartheta) = p(\theta)p(\vartheta) + (1-p(\theta))(1-p(\vartheta)), \ \ \textit{classical}. \tag{4}$$

But this equation is violated, for example, when $p(\theta+\vartheta)$=0, i.e., when $(\theta+\vartheta)$ is such that in interpretation $I^{(\theta+\vartheta)}$ "yes" ("no") means the same as the "no" ("yes") of interpretation $I^{(0)}$. Choosing $\theta = \vartheta$, then, gives us $0 = p^2 + (1-p)^2$, and this is impossible. Therefore, there must be an additional term in (4). Moreover, in our simple case we can calculate this term under the assumption

$$p(0,\theta) = p(\theta,0). \tag{5}$$

Choosing $\vartheta = -\theta$ gives us another classical equation: $1 = p^2 + (1-p)^2$, which can be rewritten as

$$1 = cos^4 f(\theta) + sin^4 f(\theta), \ \ \textit{classical}, \tag{6}$$

where the function $f(\theta)$ needs to be found. In this case, the needed additional term should be $2sin^2 f(\theta) cos^2 f(\theta)$. From this we can find

$$f(\theta) = a\theta \tag{7}$$

where a is an arbitrary real number. Indeed, from (7) it follows that

$$p(\theta + \vartheta) \equiv cos^2 a(\theta + \vartheta) = p(\theta)p(\vartheta) + (1 - p(\theta))(1 - p(\vartheta)) + interference\ \ term, \tag{8}$$

$$interference\ \ term = -2sina\theta\ \ sina\vartheta\ \ cosa\theta\ \ cosa\vartheta. \tag{9}$$

This formula gives correct results in cases $\vartheta = -\theta$, and $a\vartheta = a\theta = \pi/4$, while deviations from (7) do not.□

When $a = 1/2$, formulae (8),(9) coincide with the quantum formulae for a spin 1/2 placed on a plane, with the rotational symmetry around the axis perpendicular to this plane; θ is an angle between two axes, z and z', placed on this plane; $\Lambda^{(z)}$ is the statement "$s_z = 1/2$"; and $\bar{\Lambda}^{(z)}$ the statement "$s_z = -1/2$." From this analogy it is clear that we can introduce formal wave functions as superpositions of certain interpretations, as defined above, thus introducing a Hilbert space. (We will not do this here.) There is an essential difference, however, between our non-physical system and a quantum mechanical one. In quantum mechanics, the discreteness of spin z-projections is a direct result of $SU2$ symmetry in a three-dimensional space; such discreteness would not exist in a system with a rotational symmetry only on a plane. In our logical system, the discreteness—which is simply the discreteness of logical truth values—exists independently of symmetries.